A class of polynomials related to those of Laguerre

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ABSTRACT

We consider a class of polynomials, defined by $l_n(x) = (-1)^n L_n^{(x-n)}(x)$, which are introduced by F.G. Tricomi. We explain the role of the polynomials in asymptotics, especially in uniform expansions of a Laplace-type integral. Moreover, an asymptotic expansion of $l_n(x)$ is given for $n \to \infty$ that refines results of Tricomi and Berg.

1. Introduction

The Laguerre polynomials can be written in the form

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}, \tag{1.1}$$

where $n = 0,1,2,\cdots$, $\alpha \in \mathbb{C}$. The polynomials considered here are defined by

$$l_n(x) = (-1)^n \ L_n^{(x-n)}(x), \tag{1.2}$$

which - although closely related to the Laguerre polynomials - are essentially different from them. For instance, the degree of l_n is not n but the greatest integer $[n/2]$ in $n/2$.

The polynomials (1.2) are introduced by Tricomi [8], who used them in convergent and in asymptotic expansions of certain special functions. See also papers of Berg [1], [2], and Riekstinš [5], who too used the polynomials in asymptotic problems.

In this paper we consider a further application in the uniform asymptotic expansion of a Laplace-type integral. Furthermore we discuss the asymptotic behaviour of $l_n(x)$ as $n \to \infty$, with special attention for values of x equalling non-negative integers.

2. Uniform expansions of Laplace integrals.

We consider the integral

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt \tag{2.1}$$

for Re $z > 0$, Re $\lambda > 0$, z large, and where λ may be large as well.

When λ is restricted to a bounded set in the complex half-plane Re $z > 0$, an asymptotic expansion of (2.1) is obtained by substituting an expansion of f at $t = 0^+$. When we suppose that f is analytic at $t = 0$ (more conditions on f are given below) we obtain by using Watson's lemma (see Olver [4]) the well-known expansion

$$F_\lambda(z) \sim \sum_{s=0}^{\infty} (\lambda)_s \, a_s \, z^{-s-\lambda} \tag{2.2}$$

as $z \to \infty$ in the sector $|arg\ z| < \frac{1}{2}\pi - \delta < \frac{1}{2}\pi$. Here a_s are the coefficients in the expansion

$$f(t) = \sum_{s=0}^{\infty} a_s t^s$$

and $(\lambda)_s = \Gamma(\lambda+s)/\Gamma(\lambda)$, $s = 0,1,2,\cdots$.

The expansion (2.2) loses its asymptotic character when λ is large. For instance when $\lambda = \mathcal{O}(z)$ then the ratios of consecutive terms in (2.2) satisfy

$$\frac{a_{s+1}}{a_s} \frac{s+\lambda}{z} = \mathcal{O}(1), \quad \text{if } a_s \neq 0.$$

In [6] we modified Watson's lemma and we obtained an expansion in which large as well as small values of λ are allowed. This expansion is obtained by expanding f at $t = \mu = \lambda/z$, at which point the dominant part of the integrand of (2.1), i.e., $t^\lambda e^{-zt}$, attains its maximal value (considering real parameters for the moment). We write

$$f(t) = \sum_{s=0}^{\infty} a_s(\mu)\,(t-\mu)^s \tag{2.3}$$

and obtain by substituting this in (2.1) the formal result

$$F_\lambda(z) \sim \sum_{s=0}^{\infty} a_s(\mu)\, P_s(\lambda)\, z^{-s-\lambda}, \quad z \to \infty, \tag{2.4}$$

where

$$P_s(\lambda) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-zt} (t-\mu)^s \, dt, \quad \mu = \lambda/z. \tag{2.5}$$

The functions $P_s(\lambda)$ are polynomials in λ. They follow the recursion (which is easily obtained from (2.5))

$$P_{s+1}(\lambda) = s[P_s(\lambda) + \lambda\, P_{s-1}(\lambda)], \tag{2.6}$$

$s = 1,2,\cdots$, with initial values $P_0(\lambda) = 1$, $P_1(\lambda) = 0$. An explicit representation is obtained by expanding $(t-\mu)^s$ in powers of t. The result is

$$P_s(\lambda) = \sum_{r=0}^{s} \binom{s}{r} (\lambda)_r\, (-\lambda)^{s-r}. \tag{2.7}$$

Comparing (2.7) with (1.1), (1.2) we infer

$$P_s(\lambda) = s!\, l_s(-\lambda), \quad s = 0,1,2,\cdots,$$

which relates the polynomials $P_s(\lambda)$ with the Laguerre polynomials.

The nature of expansion (2.4) is discussed in [6], [7]. It is supposed that f is holomorphic in a connected domain Ω of the complex plane with the following conditions satisfied:

(i) the boundary $\partial\Omega$ is bounded away from $[0,\infty)$;

(ii) Ω contains a sector $S_{\alpha,\beta}$, with vertex at $t = 0$, defined by

$$S_{\alpha,\beta} = \{t \in \mathbf{C} \mid -\alpha < \arg t < \beta\},$$

where α and β are positive numbers;

(iii) $f(t) = \mathcal{O}(t^p)$ as $t \to \infty$ in $S_{\alpha,\beta}$, where p is a real number.

Under these conditions the uniformity of the expansion holds with respect to $\mu = \lambda / z$ in a closed sector, with vertex at $t = 0$, properly inside $S_{\alpha,\beta}$. Error bounds for the remainders in the expansion are also given in the cited references.

A simple example is $f(t) = 1/(1+t)$, in which event (2.1) is an exponential integral and $a_s(\mu) = (-1)^s / (1+\mu)^{s+1}$. The sector $S_{\alpha,\beta}$ is defined with $\alpha = \beta = \pi - \epsilon$ (ϵ small). We have

$$e^z E_\lambda(z) \sim \sum_{s=0}^{\infty} \frac{(-1)^s P_s(\lambda)}{(z+\lambda)^{s+1}}, \tag{2.8}$$

where $E_\lambda(z)$ is the well-known exponential integral. This example shows quite well why the uniformity with respect to λ (or to μ) holds: the degree of $P_s(\lambda)$ is $[s/2]$, and its effect is amply absorbed by the denominator in (2.8).

Another feature suggested by (2.8) is that the expansion holds for $\lambda \to \infty$, uniformly with respect to z, say $z \geqslant z_0 > 0$. This in fact is true for the general case (2.4). It has consequences on the theory of asymptotic expansions of Mellin transforms.

3. Asymptotic expansions of $l_n(x)$ as $n \to \infty$.

A generating function for the polynomials (1.2) is given by

$$e^{xz}(1-z)^x = \sum_{n=0}^{\infty} l_n(x) z^n, \quad |z| < 1, \tag{3.1}$$

where x may be any complex number; the condition on z may be dropped when $x = 0,1,2,\cdots$. Relation (3.1) is easily verified by expanding both the exponential and binomial function and by comparing the coefficients in the product with (1.1), (1.2).

Tricomi [8] investigated, among others, the asymptotic behaviour of $l_n(x)$ with n large. His final result, based on Darboux's method, can be written in the form

$$l_n(x) \sim \frac{e^x}{\Gamma(-x) n^{x+1}} \sum_{k=0}^{\infty} A_k\, n^{-k}, \tag{3.2}$$

where the coefficients A_k do not depend on n. The first few are

$$A_0 = 1, A_1 = \tfrac{3}{2}x(x+1), A_2 = x(x+1)(x+2)(27x+13)/24. \tag{3.3}$$

Observe that the right-hand side of (3.2) reduces to zero when $x = 0,1,2,\cdots$, due to the reciprocal gamma function. We cannot conclude that the polynomials reduce to zero as well, in that case; a better conclusion is that, probably, $l_n(m)$ ($m = 0,1,\cdots$) is asymptotically equal to zero with respect to the scale $\{n^{-k-x-1}\}$. For this terminology we refer to Olver [4], or to Erdélyi & Wyman [3].

From the generating function (3.1) it follows that $l_n(x)$ will exhibit a rather peculiar behaviour when x crosses non-negative integer values. Namely, the left-hand side of (3.1) is entire in z when $x = 0,1,2,\cdots$. So, for large values of n, the asymptotic behaviour of $l_n(x)$ will change considerably when x assumes these values. (In a simpler way this occurs in the binomial expansion $(1-z)^x = \sum_{n=0}^{\infty} \binom{x}{n}(-z)^n$, where the coefficients vanish identically ($n > x$) when $x = 0,1,2,\cdots$).

Berg [1] observed that for $m = 0,1,2,\cdots$ the polynomials have the asymptotic behaviour

$$l_n(m) \sim (-1)^m \frac{m^{n-m}}{(n-m)!}, \quad n \to \infty. \tag{3.4}$$

This shows indeed that the values $\{l_n(m)\}$ approach the limit 0 faster than any negative power of n.

Summarizing the above remarks we have

$$l_n(x) = \mathcal{O}(n^{-x-1}),\ x \neq 0,1,2,\cdots,$$

$$l_n(x) = \mathcal{O}(n^{-k}),\ x = 0,1,2,\ldots,\text{for any } k.$$

To give a more complete and unifying description of both these forms we look for a representation

$$l_n(x) = F_n(x) + G_n(x), \tag{3.5}$$

where $F_n(m) = 0$, $m = 0,1,2,\cdots$ and $G_n(x) = \mathcal{O}(n^{-k})$ for any k and any x; moreover, $F_n(x)$ should have Tricomi's expansion (3.2) and $G_n(m)$ that of Berg given in (3.4).

A splitting as in (3.5) is obtained by using the integral

$$l_n(x) = \frac{1}{2\pi i} \oint \frac{e^{xz}(1-z)^x}{z^{n+1}} dz, \tag{3.6}$$

which is Cauchy's representation of the coefficients in (3.1). The contour is a circle around $z = 0$ (with radius smaller than unity), or any contour that can be obtained by deformation without crossing singularities (the only candidate is $z = 1$). In (3.6) the many-valued function $(1-z)^x$ assumes its principle branch, which is real and positive for $z < 1$.

When $x \neq 0,1,\cdots$ the singular point $z = 1$ furnishes the main contribution in the asymptotic behaviour of (3.6). On the other hand, the dominant part of the integrand, which we consider to be $e^{xz}z^{-n}$, has a saddle point at $z_0 = n/x$. When we take into account contributions from $z = 1$ as well as from $z = z_0$ we are able to give a complete description of the asymptotic behaviour of $l_n(x)$.

The contour in (3.6) is deformed into the contour shown in Figure 1. We suppose, temporarily, that $x > -1$.

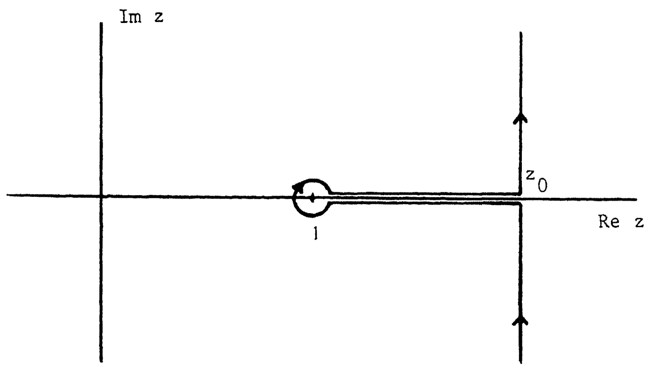

Figure 1. Contour for (3.6)

In the notation of (3.5) we choose $F_n(x)$ to be the integral around the branch cut and $G_n(x)$ the contribution over the vertical Re $z = z_0$. On the lower part of the branch cut $(1-z)^x$ is written as $(z-1)^x \exp(i\pi x)$, on the upper part as $(z-1)^x \exp(-i\pi x)$. So we obtain

$$F_n(x) = -\frac{\sin\pi x}{\pi}\int_1^{z_0} \frac{e^{xz}(z-1)^x}{z^{n+1}}\,dz = -\frac{\sin\pi x}{\pi}\,e^x \int_0^{\log z_0} u^x\,e^{-nu} f(u)du, \tag{3.7}$$

where

$$f(u) = g(u)^x,\; g(u) = \frac{e^u-1}{u}e^{e^u-1}.$$

The first coefficients in the expansion $f(u) = f_0 + f_1 u + f_2 u^2 + \cdots$ are

$$f_0 = 1, f_1 = \tfrac{3}{2}x, f_2 = (27x+13)/24.$$

So we obtain by Watson's lemma

$$F_n(x) \sim -\frac{\sin\pi x}{\pi}\frac{e^x}{n^{x+1}}\sum_{k=0}^{\infty}\frac{f_k\,\Gamma(x+k+1)}{n^k}.$$

By using the reflection formula $\Gamma(-x)\Gamma(1+x) = -\pi/\sin\pi x$ we obtain finally

$$F_n(x) \sim \frac{e^x}{\Gamma(-x)x^{n+1}}\sum_{k=0}^{\infty}\frac{f_k}{n^k}(1+x)_k,\quad n\to\infty. \tag{3.8}$$

It is easily verified that the first coefficients in (3.2) and (3.8) are the same.

Remark 3.1 The restriction on $x(x > -1)$ made earlier can be dropped by applying partial integration on the second integral in (3.7) in the form $u^x du = (x+1)^{-1}du^{x-1}$. Then a similar integral arises and the sine-function will tackle the factor $(x+1)^{-1}$ in the limit $x \to -1$.

Remark 3.2 When $x = 0,1,2,\cdots$, we can interprete (3.8) by first multiplying both sides by $\Gamma(-x)$;$\lim_{x\to m}\Gamma(-x)F_n(x)$, $m = 0,1,\cdots$, is well-defined, since now $F_n(m)$ vanishes identically. For (3.2) such an interpretation is not possible.

The expansion of the function $G_n(x)$ in (3.5) also follows from standard methods in asymptotics. Recall that $G_n(x)$ is the integral (3.6) along Re $z = z_0 = n/x$. Again we have to consider different values of $(1-z)^x$ at z_0+i0, z_0-i0. After straightforward manipulations we arrive at

$$G_n(x) = \frac{e^n z_0^{x-n}}{\pi}\mathrm{Re}\{e^{-i\pi x}\int_0^{\infty} e^{in\tau - n\ln(i+i\tau)}\frac{(1+i\tau-1/z_0)^x}{1+i\tau}\,d\tau\}. \tag{3.9}$$

To obtain a first apporximation we replace $i\tau - \ln(1+i\tau)$ by the first non-vanishing term of its Maclaurin expansion, i.e., $-\frac{1}{2}\tau^2$, and $(1+i\tau-1/z_0)^x/(1+i\tau)$ by unity. Then we have

$$G_n(x) \sim (2\pi n)^{-\frac{1}{2}} e^n (n/x)^{x-n}\cos(\pi x), \tag{3.10}$$

which for $x = m = 0,1,2,\cdots$ agrees with the right-hand side of (3.4), when we replace the factorial by its Stirling approximation. Higher approximations can easily be obtained from (3.9), but will not be given here.

Remark 3.3. The cosine-term in (3.10) does not appear in all higher approximations of $G_n(x)$.

References

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